The integral cohomology rings of some p-groups

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1. Introduction.

We determine the integral cohomology rings of an infinite family of p-groups, for odd primes p, with cyclic derived subgroups. Our method involves embedding the groups in a compact Lie group of dimension one, and was suggested independently by P. H. Kropholler and J. Huebschmann. This construction has also been used by the author to calculate the mod-p cohomology of the same groups and by B. Moselle to obtain partial results concerning the mod-p cohomology of the extra special p-groups [7], [9].

2 The method and the groups.

Given a finite group G and a central cyclic subgroup C, we fix an embedding of C into S^1, and define a Lie group $\widetilde{G}$ as the product of S^1 and G amalgamating C, that is

$$\widetilde{G} = \mathrm{S}^1 \times G/\{(c^{-1}, c) : c \in C\}$$

Then we have a commutative diagram:

$$\begin{array}{ccccc} C & \longrightarrow & G & \longrightarrow & Q \\ \downarrow & & \downarrow & & \downarrow \\ \mathrm{S}^1 & \longrightarrow & \widetilde{G} & \longrightarrow & Q. \end{array}$$

If M is a G-module on which C acts trivially, we may consider M as a $\widetilde{G}$-module by letting S^1 act trivially, and the Lyndon-Hochschild-Serre spectral sequence for the second extension is often simpler than that for the first. To find $H^*(\mathrm{B}G; M)$, given $H^*(\mathrm{B}\widetilde{G}; M)$, we use the Serre spectral sequence of the fibration

$$\mathrm{S}^1/C \cong \widetilde{G}/G \longrightarrow \mathrm{B}G \longrightarrow \mathrm{B}\widetilde{G}.$$

This spectral sequence has $E_2^{i,j} = 0$ for $j > 1$, so the only possible non-zero differential is d_2. The above was first suggested to the author by P. Kropholler. A similar idea occurs in J. Huebschmann's papers [5] and [6]. In the case where M is a commutative ring on which G acts trivially, it appears that we may obtain another filtration of $H^*(\mathrm{B}G; M)$ by examining the Eilenberg-Moore spectral sequence for the pullback square:

$$\begin{array}{ccc} \mathrm{B}G & \longrightarrow & \mathrm{B}\widetilde{G} \\ \downarrow & & \downarrow \\ \{*\} & \longrightarrow & \mathrm{B}\widetilde{G}/G, \end{array}$$

but it can be shown that the two filtrations are identical. These spectral sequences are just alternative ways to view the Gysin sequence for the S^1-bundle $\mathrm{B}G$ over $\mathrm{B}\widetilde{G}$.

It is possible for non-equivalent extensions of C by Q to yield equivalent extensions of S^1 by Q. In fact this happens if and only if their extension classes in $H^2(Q; C)$ map to

the same element of $H^3(Q;\mathbb{Z})$ under the Bockstein associated with the coefficient sequence $\mathbb{Z} \rightarrowtail \mathbb{Z} \twoheadrightarrow C$.

The groups we shall consider are central extensions of $C_{p^{n-2}}$ by $C_p \oplus C_p$ where p is an odd prime, and $n \geq 3$. They may be presented as

$$P(n) = \langle A, B, C \mid A^p = B^p = C^{p^{n-2}} = [A, C] = [B, C] = 1, [A, B] = C^{p^{n-3}} \rangle.$$

We shall let $\widetilde{P}$ be the corresponding central extension of S^1 by $C_p \oplus C_p$, that is the group obtained from $\mathrm{S}^1 \times P(n)$ by amalgamating the subgroup of $P(n)$ generated by $\langle C \rangle$ and the $C_{p^{n-2}}$ subgroup of S^1. There are four central extensions of $C_{p^{n-2}}$ by $C_p \oplus C_p$; two abelian ones, $P(n)$, and a metacyclic group $M(n)$ containing a cyclic subgroup of index p. This may be checked by verifying that the action of $\mathrm{Aut}(C_p \oplus C_p)$ on $H^2(\mathrm{B}(C_p \oplus C_p); C_{p^{n-2}})$ has only four orbits, and then explicitly constructing four non-isomorphic groups. There are however only two central extensions of S^1 by $C_p \oplus C_p$; the direct product, which is abelian, and $\widetilde{P}$ which is not. This follows from the fact that $\mathrm{Aut}(C_p \oplus C_p)$ acts transitively on the non-zero elements of $H^3(\mathrm{B}(C_p \oplus C_p); \mathbb{Z})$, which may be identified with $H^2(\mathrm{B}(C_p \oplus C_p); \mathrm{S}^1)$ via the Bockstein associated with the coefficient sequence $\mathbb{Z} \rightarrowtail \mathbb{R} \twoheadrightarrow \mathrm{S}^1$. Hence we see that $\mathrm{B}M(n)$ is also an S^1-bundle over $\mathrm{B}\widetilde{P}$, and in fact $H^*(\mathrm{B}M(n); \mathbb{Z})$ could easily be determined from the results of this paper. This cohomology ring has already been calculated using other methods [11].

3. Calculations.

We now begin our calculation of $H^*(\mathrm{B}\widetilde{P})$ by examining the spectral sequence with integer coefficients for $\widetilde{P}$ considered as an extension of S^1 by $C_p \oplus C_p$. The E_2 page is readily seen to be generated by elements $\alpha, \beta \in E_2^{2,0}$, $\gamma \in E_2^{3,0}$ and $\tau \in E_2^{0,2}$ subject only to the relations $p\alpha = p\beta = 0$, $p\gamma = 0$ and $\gamma^2 = 0$. Note that τ has infinite order. Since $E_2^{i,j}$ is trivial for j odd, we see that all the even differentials must vanish. The behaviour of the differentials is summarised in the following lemma.

Lemma 1. *In the above spectral sequence there are exactly two non-zero differentials, d_3 and d_{2p-1}. $d_3(\tau)$ is a non-zero multiple of γ, and E_4 is generated by the classes of the elements $\alpha, \beta, p\tau, \ldots, p\tau^{p-1}, \tau^p$ and $\tau^{p-1}\gamma$. All of these generators are universal cycles except for $\tau^{p-1}\gamma$, which is mapped by d_{2p-1} to a non-zero multiple of $\alpha^p\beta - \beta^p\alpha$. The E_∞ page is generated by the elements $\alpha, \beta, p\tau, \ldots, p\tau^{p-1}, \tau^p$ subject only to the relations they satisfy as elements of E_2, and the relation $\alpha^p\beta = \beta^p\alpha$.*

Proof. The derived subgroup of $\widetilde{P}$ consists of the subgroup of its central S^1 of order p, so there can be no homomorphism from $\widetilde{P}$ to S^1 that restricts to an isomorphism from the centre to S^1. It follows by considering the natural isomorphism $H^2(\mathrm{B}G;\mathbb{Z}) \cong \mathrm{Hom}(\mathrm{B}G, \mathrm{S}^1)$ that the element τ cannot survive to E_∞, so we must have $d_3(\tau)$ a non-zero multiple of γ. This determines d_3 completely. It may be checked that E_4 is isomorphic to the subring of E_2 generated by $\alpha, \beta, p\tau, \ldots, p\tau^{p-1}, \tau^p$ and $\tau^{p-1}\gamma$. All these elements must be universal cycles, with the possible exception of $\tau^{p-1}\gamma$, because the groups in which their images under d_n lie are already trivial. The only remaining potentially non-zero differential is $d_{2p-1}(\tau^{p-1}\gamma)$. To complete this proof it suffices to show that in the E_∞ page the relation $\alpha^p\beta = \beta^p\alpha$ must hold.

Let Q be the quotient of $\widetilde{P}$ by its S^1 subgroup, and take generators α', β' for $H^2(\mathrm{B}Q;\mathbb{Z})$ and γ' for $H^3(\mathrm{B}Q;\mathbb{Z})$. The statement that γ does not survive to E_∞ in the spectral sequence is equivalent to the statement that γ' is mapped to zero by the inflation map

from Q to $\widetilde{P}$. Now we calculate $\phi(\gamma')$, where ϕ is the integral cohomology operation $\delta_p \mathrm{P}^1 \pi_*$, where π_* is the map induced by the change of coefficients from $\mathbb{Z}$ to $\mathbb{F}_p$, P^1 is a reduced power, and δ_p is the Bockstein for the sequence $\mathbb{Z} \rightarrowtail \mathbb{Z} \twoheadrightarrow \mathbb{F}_p$. Taking $y, y' \in H^1(\mathrm{B}Q; \mathbb{F}_p)$ such that $\delta_p(y) = \alpha'$, and $\delta_p(y') = \beta'$, we see that

$$\phi(\gamma') = \delta_p \mathrm{P}^1 \pi_*(\gamma') = \delta_p \mathrm{P}^1(\beta_p(y)y' - \beta_p(y')y) = \delta_p(\beta_p(y)^p y' - \beta_p(y')^p y) = \alpha'^p \beta' - \beta'^p \alpha'.$$

It follows that

$$\alpha^p \beta - \beta^p \alpha = \inf(\alpha'^p \beta' - \beta'^p \alpha') = \inf(\phi(\gamma')) = \phi \inf(\gamma') = 0.$$

■

We note that the result on d_{2p-1} could also be considered as a case of an integral version of Kudo's transgression theorem. We are now ready to state our theorem on $H^*(\widetilde{P})$.

Theorem 2. *Let p be an odd prime, and let $\widetilde{P}$ be the group defined above. Then $H^*(\widetilde{P}; \mathbb{Z})$ is generated by elements α,β,$\chi_1, \ldots, \chi_{p-1}$, ζ, with*

$$\deg(\alpha) = \deg(\beta) = 2 \quad \deg(\chi_i) = 2i \quad \deg(\zeta) = 2p,$$

subject to the following relations:

$$p\alpha = p\beta = 0$$

$$\alpha^p \beta = \beta^p \alpha$$

$$\alpha\chi_i = \begin{cases} 0 \\ -\alpha^p \end{cases} \qquad \beta\chi_i = \begin{cases} 0 & \text{for } i < p-1 \\ -\beta^p & \text{for } i = p-1 \end{cases}$$

$$\chi_i \chi_j = \begin{cases} p\chi_{i+j} & i+j < p \\ p^2\zeta & i+j = p \\ p\zeta\chi_{i+j-p} & p < i+j < 2p-2 \\ p\zeta\chi_{p-2} + \alpha^{2p-2} + \beta^{2p-2} - \alpha^{p-1}\beta^{p-1} & i = j = p-1 \end{cases}$$

Chern classes of representations of $\widetilde{P}$ generate the whole ring. An automorphism of $\widetilde{P}$ sends χ_i to χ_i (resp. $(-1)^i\chi_i$) and ζ to ζ (resp. $-\zeta$) if it fixes (resp. reverses) S^1. The

effect of an automorphism on α, β may be determined from their definition. Considered as elements of $\mathrm{Hom}(\widetilde{P}, \mathrm{S}^1)$, *$\alpha$ has kernel $\langle \mathrm{S}^1, B\rangle$ and sends A to $e^{2\pi i/p}$, and β has kernel $\langle \mathrm{S}^1, A\rangle$ and sends B to $e^{2\pi i/p}$. If we let H be the subgroup generated by B and elements of* S^1 *we may define*

$$\chi_i = \begin{cases} \mathrm{Cor}_H^{\widetilde{P}}(\tau'^i) & \text{for } i < p-1 \\ \mathrm{Cor}_H^{\widetilde{P}}(\tau'^{p-1}) - \alpha^{p-1} & \text{for } i = p-1 \end{cases}$$

where τ' is any element of $H^2(H;\mathbb{Z})$ restricting to S^1 *as the generator τ. Similarly, $\zeta = c_p(\rho)$, where ρ is an irreducible representation of $\widetilde{P}$ restricting to* S^1 *as p copies of the representation ξ with $c_1(\xi) = \tau$.*

Proof. First we note that in the E_∞ page of the above spectral sequence all the group extensions that we need to examine are extensions of finite groups by the infinite cyclic group, so are split. The elements α and β defined in the statement above clearly yield generators for $E_\infty^{2,0}$, and the relations between them are exactly the relations that hold between the corresponding elements in the spectral sequence. Let β' in $H^2(\mathrm{B}H)$ be the restriction to H of β, and take any choice of τ' as in the statement. We may show by considering β' and τ' as homomorphisms from H to S^1 that conjugation by A^i induces the map on $H^2(\mathrm{B}H)$ that fixes β' and sends τ' to $\tau' - i\beta'$. Now applying the formula for $\mathrm{Res}_K^G \mathrm{Cor}_H^G$ (see for example [4]) it follows that χ_i restricts to S^1 as $p\tau^i$, so yields a generator for $E_\infty^{0,2i}$.

Any irreducible representation of $\widetilde{P}$ has degree 1 or p, because $\widetilde{P}$ has an abelian subgroup of index p. Let ρ be the representation of $\widetilde{P}$ induced from a 1-dimensional representation of H with first Chern class τ'. ρ restricts to S^1 as p copies of the representation with first Chern class τ, so its total Chern class restricts to S^1 as $(1+\tau)^p$, and so $c_p(\rho)$ yields a generator for $E_\infty^{0,2p}$, and $c_i(\rho) = 1/p\binom{p}{i}\chi_i + P_i(\alpha, \beta)$ for some polynomial P_i. We shall show later that $P_i = 0$.

The restriction to H of α is trivial, so by Fröbenius reciprocity

$$\alpha \mathrm{Cor}_H^{\widetilde{P}}(\tau'^i) = \mathrm{Cor}_H^{\widetilde{P}}(\mathrm{Res}_H^{\widetilde{P}}(\alpha)\tau'^i) = 0,$$

and the expressions given for $\alpha\chi_i$ follow. By calculating $\alpha(\beta\chi_i) = \beta(\alpha\chi_i)$, we may deduce that $\beta\chi_i = 0$ for $i < p-1$ and $\beta\chi_{p-1} = \lambda(\alpha^{p-1}\beta - \beta^p) - \alpha^{p-1}\beta$ for some scalar λ. To show that $\lambda = 1$ we use the restriction map to H, and the formula for corestriction followed by restriction.

$$\begin{aligned}\mathrm{Res}_H^{\widetilde{P}}(\beta\chi_{p-1}) &= \beta' \sum_{i=0}^{p-1} (\tau' + i\beta')^{p-1} \\ &= \beta' \sum_{j=0}^{p-1} \tau'^{p-1-j}\beta'^j \sum_{i=0}^{p-1} i^j\end{aligned}$$

Newton's formula tells us that

$$\sum_{i=1}^{p-1} i^j \equiv \begin{cases} 0 \ \ (p) & \text{for } j \not\equiv 0 \ \ (p-1) \\ 1 \ \ (p) & \text{for } j \equiv 0 \ \ (p-1) \end{cases}$$

so $\mathrm{Res}_H^{\widetilde{P}}(\beta\chi_{p-1}) = -\beta'^p$, and the required relation follows.

We now know $\mathrm{Res}_{\mathrm{S}^1}^{\widetilde{P}}(\chi_i\chi_j)$, $\alpha\chi_i\chi_j$, and $\beta\chi_i\chi_j$, which together imply the relations given for $\chi_i\chi_j$. To complete the proof of the theorem we must determine the effect of automorphisms of $\widetilde{P}$ on the χ_i. We know that an automorphism sends $c_i(\rho)$ to itself or $(-1)^i$ times itself depending whether or not it reverses the sense of S^1, so it will suffice to show that $\chi_i = 1/p\binom{p}{i}c_i(\rho)$. The character of ρ is zero except on S^1, so if θ is a 1-dimensional representation of $\widetilde{P}$ restricting trivially to S^1, then $\rho \otimes \theta$ is isomorphic to ρ. If we apply the formula expressing $c.(\rho \otimes \theta)$ in terms of $c.(\rho)$ and $c.(\theta)$ (see [3]) we obtain

$$c_i(\rho) = c_i(\rho \otimes \theta) = \sum_{j=0}^{i} \binom{p-i+j}{j} c_1(\theta)^j c_{i-j}(\rho).$$

and hence inductively

$$c_i(\rho)c_1(\theta) = \begin{cases} 0 & \text{for } i < p-1 \\ -c_1(\theta)^p & \text{for } i = p \end{cases} .$$

Since α and β are possible values for $c_1(\theta)$ the required result follows. We may show inductively that χ_i is in the subring generated by Chern classes because χ_1 is, and $\chi_1\chi_{i-1}$, $1/p\binom{p}{i}\chi_i$ are coprime multiples of χ_i. ∎

We are now ready to state our theorem on the integral cohomology of $\mathrm{B}P(n)$.

Theorem 3. *Let p be an odd prime and let $P(n)$ be as defined above. Then $H^*(\mathrm{B}P(n);\mathbb{Z})$ is generated by elements $\alpha, \beta, \mu, \nu, \chi_1, \ldots, \chi_{p-1}, \zeta$, with*

$$\deg(\alpha) = \deg(\beta) = 2 \quad \deg(\mu) = \deg(\nu) = 3 \quad \deg(\chi_i) = 2i \quad \deg(\zeta) = 2p$$

subject to the following relations:

$$p\alpha = p\beta = 0 \quad p\mu = p\nu = 0 \quad p^{n-3}\chi_1 = 0 \quad p^{n-2}\chi_i = 0 \quad p^{n-1}\zeta = 0$$

$$\alpha\mu = \beta\nu$$

$$\alpha^p\beta = \beta^p\alpha \quad \alpha^p\mu = \beta^p\nu$$

$$\alpha\chi_i = \begin{cases} 0 \\ -\alpha^p \end{cases} \qquad \beta\chi_i = \begin{cases} 0 & \text{for } i < p-1 \\ -\beta^p & \text{for } i = p-1 \end{cases}$$

$$\mu\chi_i = \begin{cases} 0 \\ -\beta^{p-1}\mu \end{cases} \qquad \nu\chi_i = \begin{cases} 0 & \text{for } i < p-1 \\ -\alpha^{p-1}\nu & \text{for } i = p-1 \end{cases}$$

$$\chi_i\chi_j = \begin{cases} p\chi_{i+j} & i+j < p \\ p^2\zeta & i+j = p \\ p\zeta\chi_{i+j-p} & p < i+j < 2p-2 \\ p\zeta\chi_{p-2} + \alpha^{2p-2} + \beta^{2p-2} - \alpha^{p-1}\beta^{p-1} & i = j = p-1 \end{cases}$$

$$\mu\nu = \begin{cases} 0 & \text{for } n > 3 \\ \lambda\chi_3 & \text{for } n = 3,\ p > 3, \lambda \in \mathbb{Z}_p^\times \\ 3\lambda\zeta & \text{for } n = 3,\ p = 3,\ \lambda = \pm 1 \end{cases}$$

Chern classes of representations of $P(n)$ generate $H^{\mathrm{even}}(\mathrm{B}P(n);\mathbb{Z})$. Under an automorphism of $P(n)$ which restricts to the centre as $C \mapsto C^j$, χ_i is mapped to $j^i\chi_i$, and ζ is mapped to $j^p\zeta$. The effect of automorphisms on α and β is determined by the natural isomorphism $H^2(\mathrm{B}G;\mathbb{Z}) \cong \mathrm{Hom}(G, \mathbb{R}/\mathbb{Z})$, under which

$$\begin{array}{lll} \alpha : A \mapsto 1/p & \beta : A \mapsto 0 & \chi_1 : A \mapsto 0 \\ B \mapsto 0 & B \mapsto 1/p & B \mapsto 0 \\ C \mapsto 0 & C \mapsto 0 & C \mapsto 1/p^{n-3}. \end{array}$$

An automorphism of $P(n)$ which sends α to $n_1\alpha + n_2\beta$, β to $n_3\alpha + n_4\beta$ and restricts to the centre as $C \mapsto C^j$ sends μ to $j(n_4\mu + n_3\nu)$ and ν to $j(n_2\mu + n_1\nu)$. If γ' in $H^2(\mathrm{B}\langle B, C\rangle;\mathbb{Z})$ is such that it maps to the following element of $\mathrm{Hom}(\langle B, C\rangle, \mathbb{R}/\mathbb{Z})$

$$\begin{array}{l} \gamma' : B \mapsto 0 \\ \quad C \mapsto 1/p^{n-2} \end{array},$$

then χ_i is defined as follows:

$$\chi_i = \begin{cases} \mathrm{Cor}_{\langle B,C\rangle}^{P(n)}(\gamma'^i) & \text{for } i < p-1 \\ \mathrm{Cor}_{\langle B,C\rangle}^{P(n)}(\gamma'^{p-1}) - \alpha^{p-1} & \text{for } i = p-1. \end{cases}$$

These are, up to scalar multiples, equal to $c_i(\rho)$, where ρ is a p-dimensional irreducible representation of $P(n)$, whose restriction to $\langle C\rangle$ is a sum of p copies of the representation θ, with $c_1(\theta) = \mathrm{Res}_{\langle C\rangle}^{\langle B,C\rangle}(\gamma')$. In fact, $c_i(\rho) = 1/p\binom{p}{i}\chi_i$. Also, we may define $\zeta = c_p(\rho)$.

Proof. We examine the spectral sequence for $\mathrm{B}P(n)$ as an S^1-bundle over $\mathrm{B}\widetilde{P}$. $E_2^{*,0}$ is isomorphic to $H^*(\mathrm{B}P(n);\mathbb{Z})$ and $E_2^{*,*}$ is freely generated by $E_2^{*,0}$ and an element ξ of infinite order in $E_2^{0,1}$. We know that $H^2(\mathrm{B}P(n)) \cong \mathrm{Hom}(P(n),\mathrm{S}^1) \cong C_{p^{n-3}} \oplus C_p \oplus C_p$, so $d_2(\xi)$ must be $\pm p^{n-3}\chi_1$. If we wanted to calculate the cohomology of the metacyclic groups $M(n)$ described above, the differential in this spectral sequence would send ξ to $\pm p^{n-3}\chi_1 + \gamma$ for some non-zero γ in $\langle \alpha, \beta\rangle$. It is now easy to see that E_∞ is generated by the elements $\alpha, \beta, \mu = \beta\xi, \nu = \alpha\xi, \chi_1, \ldots, \chi_{p-1}$ and ζ subject to the relations they satisfy as elements of $E_2^{*,*}$ together with $p^{n-3}\chi_1 = 0$, $p^{n-2}\chi_i = 0$, and $p^{n-1}\zeta = 0$. For each m, the filtration of $H^m(\mathrm{B}P(n))$ given by the E_∞ page is trivial, so we may use the same symbols to denote elements of $H^m(\mathrm{B}P(n))$, and the relations that hold in E_∞ determine all the relations that hold in $H^m(\mathrm{B}P(n))$ except for the product of the two odd dimensional generators.

We know that $p\mu\nu = 0$, and the relation $\alpha\mu = \beta\nu$ implies that $\alpha\mu\nu = \beta\mu\nu = 0$, and so $\mu\nu$ must be a multiple of $p^{n-3}\chi_3$ for $p \geq 5$ (resp. 3ζ for $p = 3$). Note that these elements restrict to zero on all proper subgroups of $P(n)$. In the case of $P(3)$, Lewis [8] shows that $\mu\nu$ is not zero by considering the spectral sequence for $P(n)$ considered as an extension of a maximal subgroup by C_p. A similar method will work in general, but we offer an alternative proof that involves expressing μ and ν as Bocksteins of elements of $H^2(\mathrm{B}P(n);\mathbb{F}_p)$. This proof is contained in lemma 4 and corollary 5.

The effect of automorphisms on χ_i and ζ is easily seen to be as claimed from their alternative definitions as Chern classes. To determine the effect of automorphisms on μ

and ν, we note that an automorphism of $P(n)$ restricting to the centre as $C \mapsto C^j$ extends to an endomorphism of $\widetilde{P}$ which wraps the central circle j times around itself, so induces a map of the above spectral sequence to itself sending ξ to $j\xi$. This completes the proof of theorem 3 modulo lemma 4 and its corollary. ∎

We now examine the spectral sequence with $\mathbb{F}_p$ coefficients for the central extension $C_{p^{n-2}} \rightarrowtail P(n) \twoheadrightarrow C_p \oplus C_p$. Take generators so that $H^*(\mathrm{B}C_p \oplus C_p; \mathbb{F}_p) \cong \mathbb{F}_p[x, x'] \otimes \Lambda[y, y']$, where $\beta_p(y) = x$, $\beta_p(y') = x'$, and $H^*(\mathrm{B}C_{p^{n-2}}; \mathbb{F}_p) \cong \mathbb{F}_p[t] \otimes \Lambda[u]$, where $\beta_p(u) = t$ for $n = 3$ (resp. $\beta_p(u) = 0$ for $n \geq 4$). Then the E_2 page is isomorphic to $\mathbb{F}_p[x, x', t] \otimes \Lambda[y, y', u]$, and the first two differentials are as described in the following lemma.

Lemma 4. *With notation as above, identify the elements x, x', y, y' in the spectral sequence with their images in $H^*(\mathrm{B}P(n); \mathbb{F}_p)$ under the inflation map.*

1) *Let $n \geq 4$. Then d_2 is trivial, and $d_3(t)$ is a non-zero multiple of $xy' - x'y$. The set $\{x, x', yy', u'y, u'y'\}$ is a basis for $H^2(\mathrm{B}P(n))$, where u' is any element of $H^1(\mathrm{B}P(n))$ restricting to $C_{p^{n-2}}$ as u.*

2) *Let $n = 3$. Then $d_2(u)$ is a non-zero multiple of yy', $d_2(t) = 0$, and E_3 is generated by $y, y', x, x', [uy], [uy']$ and t subject to the relation $yy' = 0$ and those implied by the relations in E_2. In particular $[uy]y' = -[uy']y$ but this element is non-zero. As in the case $n \geq 4$, $d_3(t)$ is a non-zero multiple of $xy' - x'y$. Let Y, Y' be elements of $H^2(\mathrm{B}P(n))$ such that $\{x, x', Y, Y'\}$ is a basis for $H^2(\mathrm{B}P(3))$, and let $X = \beta_p(Y)$, $X' = \beta_p(Y')$. Then $\{yY', xy, xy', x'y', X, X'\}$ is a basis for $H^3(\mathrm{B}P(3))$ and $\{xX, xX', x'X', x^2y, x^2y', xx'y', x'^2y', YX'\}$ is a basis for $H^5(\mathrm{B}P(3))$.*

Proof. 1). In this case H^1 has order p^3, so u must survive. The element $xy' - x'y$ is the image under π_* of a generator for $H^3(\mathrm{B}(C_p \oplus C_p); \mathbb{Z})$, so must be killed by some differential. We have already shown that it cannot be killed by d_2, so the only possibility is that t survives until E_3 and kills it. The rest of the statement follows easily.

2). In this case H^1 has order p^2, so $d_2(u)$ must be non-zero. It is true in general that

if G is a central extension of C_p by Q, then in the corresponding spectral sequence with $\mathbb{F}_p$ coefficients $d_2 : E_2^{0,1} \to E_2^{2,0}$ must kill the extension class. This follows by naturality, since one may regard the extension class as defining a homotopy class of maps from $\mathrm{B}Q$ to $K(C_p, 2)$ such that $\mathrm{B}G$ is the $\mathrm{B}C_p$-bundle induced by the path-loop fibration over $K(C_p, 2)$. Since all subgroups of $P(3)$ of order p^2 are copies of $C_p \oplus C_p$, the extension class of $P(3)$ must restrict to zero on all cyclic subgroups, so must be a multiple of yy'. The transgression commutes with the Bockstein so $d_2(t) = 0$ and $d_3(t) = \beta_p d_2(u)$.

Given the values of these differentials it is routine to compute the E_4 page of the spectral sequence. If we write $E_r^n = \bigoplus_{i+j=n} E_r^{i,j}$, then $\{[uy], [uy'], x, x'\}$ forms a basis for $E_4^2 = E_\infty^2$, and $\{[ty], [ty'], [uy]y', xy, xy', x'y'\}$ forms a basis for $E_4^3 = E_\infty^3$. The spectral sequence operation ${}_F\beta$ introduced by Araki [2] and Vasquez [12] maps $[uy]$ to $[ty]$ and $[uy']$ to $[ty']$, so if Y and Y' are chosen to yield the generators for $E_4^{1,1}$ their Bocksteins yield generators for $E_4^{1,2}$. A basis for E_4^5 is given by the eight elements of the statement, which we know to be universal cycles, and the elements $[t^2y]$, $[t^2y']$. E_4^4 consists of universal cycles, and the universal coefficient theorem tells us that H^5 has order p^8, so $[t^2y]$ and $[t^2y']$ cannot be universal cycles. ■

Corollary 5. *In $H^*(\mathrm{B}P(n); \mathbb{Z})$ the product $\mu\nu$ is non-zero if and only if $n = 3$.*

Proof. In the notation of lemma 4 it suffices to determine $\delta_p(u'y)\delta_p(u'y')$ in the case $n \geq 4$, and $\delta_p(Y)\delta_p(Y')$ in the case $n = 3$. In the case when $n = 3$,

$$\delta_p(Y)\delta_p(Y') = \delta_p(Y\beta_p(Y')) = \delta_p(YX').$$

The kernel of $\delta_p : H^5(\mathrm{B}P(3); \mathbb{F}_p) \to H^6(\mathrm{B}P(3); \mathbb{Z})$ is equal to $\pi_*(H^5(\mathrm{B}P(3); \mathbb{Z}))$, which is generated by xX, xX' and $x'X'$, so by lemma 4 $\delta_p(YX')$ is non-zero.

In the case when $n = 4$, $H^i(\mathrm{B}P(4); \mathbb{Z})$ has exponent p for $i = 2, 3$, so π_* is injective from these groups, and $\ker \beta_p : H^2(\mathrm{B}P(4)) \to H^3(\mathrm{B}P(4))$ is equal to $\beta_p(H^1(\mathrm{B}P(4)))$. $\beta_p(yy') = xy' - x'y = 0$, so we may choose the element u' in lemma 4 so that $\beta_p(u') = \lambda yy'$

for some non-zero λ. Then we have

$$\delta_p(u'y)\delta_p(u'y') = \delta_p(u'y\beta_p(u'y')) = \delta_p(u'y(\lambda yy'y' - u'x')) = 0.$$

The case when $n \geq 5$ is similar but simpler, since u' may be chosen so that $\delta_p(u') = p^{n-4}\chi_1$, which implies that $\beta_p(u') = 0$. ■

Remarks. Theorem 3 contains independent proofs of Thomas' result that the even degree subring of $H^*(\mathrm{B}P(n);\mathbb{Z})$ is generated by Chern classes [10], and Lewis' calculation of $H^*(\mathrm{B}P(3);\mathbb{Z})$. Our notation differs slightly from that of Lewis. We have renumbered the generators χ_i (note that χ_1 vanishes for $n = 3$). Also our χ_{p-1} and Lewis' χ_{p-2} are related by the formula

$$\chi_{p-2}^{\text{Lewis}} = \chi_{p-1} + \alpha^{p-1} + \beta^{p-1}.$$

Our result disagrees with that of AlZubaidy [1].

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